

AN IUH EQUATION BASED ON ENTROPY THEORY

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ABSTRACT. *The instantaneous unit hydrograph (IUH) is a commonly used method for computing surface runoff hydrographs from small watersheds. Assuming travel time as a random variable, a general equation for IUH is derived using the entropy theory. This equation specializes into several well-known equations, such as the gamma distribution, Lienhard distribution, and Nakagami-m distribution, to name but a few. The general equation has three parameters, two of which are based on specified information (or constraints) on travel time, and the third parameter is an exponent that can also be determined from the specified values of travel time. In this study, the derived equation is tested on two small agricultural experimental watersheds. Surface runoff hydrographs computed using the derived IUH equation are found to be in satisfactory agreement with observed surface runoff hydrographs.*

Keywords. *Entropy, Gamma distribution, Instantaneous unit hydrograph, Lagrange multipliers, Principle of maximum entropy, Shannon entropy.*

Since the development of the unit hydrograph concept by Sherman (1932) and the subsequent development of the unit hydrograph theory by Nash (1957, 1958, 1959) and Dooge (1959, 1973), linear systems based methods (Singh, 1988, 1989) have been commonly employed for modeling rainfall-runoff response, flow routing, sediment routing, and pollutant routing. For computing flood hydrographs, the unit hydrograph is still one of the commonly used models. The unit hydrograph, or more appropriately the instantaneous unit hydrograph (IUH), represents the response of a linear time-invariant system, i.e., the watershed is assumed to be linear, meaning the principle of superposition applies; and time invariant, meaning the kernel remains independent of time. This implies that the IUH represents the response of the watershed in terms of surface runoff due to an instantaneous excess rainfall of unit volume.

Surface runoff or discharge at the outlet or mouth of a watershed in response to a given rainfall event is a function of rainfall intensity and duration, infiltration, antecedent soil moisture, and watershed characteristics. The IUH concept assumes that the excess rainfall occurs uniformly over the area for which the IUH is derived. This area can be the entire watershed if it is small (less than 250 km²) or a subwatershed (a portion) therein if it is large (greater than or equal to 2500 km²). In this manner, the IUH can be applied to large watersheds. The assumption of spatial uniformity of infiltration is implied in the concept of rainfall excess and in turn in spatial uniformity of antecedent soil moisture, and for that matter this assumption also applies to other abstractions, such as evaporation, interception, and depression and detention stor-

age. This assumption is only an approximation and can be pesky in some cases.

Keeping the above assumptions in mind, surface runoff (or discharge) as a function of time, $Q(t)$, is obtained by convoluting the IUH, $h(t)$, with the rainfall excess hyetograph, $I(t)$, as:

$$Q(t) = \int_0^t h(t - \tau)I(\tau)d\tau \quad (1a)$$

The IUH has dimensions of 1/T if surface runoff is expressed in units of L/T or dimensions of L²/T if runoff is expressed in units of L³/T, where L and T represent the length and time dimensions. For computation, equation 1a can also be expressed in discrete form as:

$$Q_j = \sum_{i=1}^j h_{j-i}I_i\Delta t \quad (1b)$$

where Q_j is the surface runoff at time j , h_{j-i} is the IUH ordinate at time $j-i$, I_i is the excess rainfall intensity at time i , and Δt is the time interval for computation over which the excess rainfall intensity is assumed constant. The total duration of surface runoff will be the sum of the duration of excess rainfall and the duration of the IUH.

Fundamental to determining $Q(t)$ for a given $I(t)$ is the determination of the IUH (Nash, 1957; Dooge, 1973). There is a multitude of ways in which the IUH has been derived (Dooge, 1973; Singh, 1988). Some of these are entirely empirical, and no mathematical equation of the IUH is given. Examples of such methods are the Laplace transform method, Fourier transform method, triangular unit hydrograph method, and time area method (Dooge, 1973; Singh, 1988). Synthetic unit hydrograph methods are empirical methods wherein unit hydrograph (UH) characteristics, such as peak, time to peak or centroid, and total duration of the UH are related to watershed characteristics, such as area, length, slope, and distance to the centroid (Snyder, 1938; Taylor and Schwarz, 1952; Gray, 1961; Crowley, 1980). From these characteristics, the IUH is constructed. Other methods are

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based on assumed algebraic equations or probability distribution functions and their fit to data (Yue et al., 2002; Bhunya et al., 2004; Jain et al., 2005; Nadarajah, 2007; Rai et al., 2008). However, such equations lack a physical basis, and therefore these are also entirely empirical. Another way is the use of linear systems theory (Nash, 1957; Dooge, 1959) employing the spatially lumped form of a continuity equation and a storage-discharge relation, which can be construed as an approximation of the momentum conservation or the energy conservation equation. The linear systems theory represents a watershed by linear reservoirs and channels. Depending on the architecture of these reservoirs and channels, an IUH equation is derived. This architecture is determined somewhat arbitrarily, although attempts have been made to derive it from watershed geomorphology (Singh, 1988, 1996). However, an objective way of determining the architecture is still not known. Another popular method for deriving the unit hydrograph is the coupling of linear systems theory with the laws of geomorphology (Rodriguez-Iturbe and Valdes, 1979; Gupta and Waymire, 1983).

Employing the Boltzmann statistics and discrete-time representation, Lienhard (1964) developed an analytical form of the dimensionless unit hydrograph and showed that it was almost independent of the watershed properties. Although Lienhard (1964) and Lienhard and Meyer (1967) did not explicitly use the Shannon entropy theory, it is clear that they were the first to introduce the concept of entropy to the unit hydrograph literature. Their work lays the groundwork for application of the entropy theory and motivated the present study.

The objective of this article, therefore, is to derive a general IUH equation using the entropy theory. This general IUH equation can specialize into some of the popular forms used in hydrology. The advantage of using the entropy theory is twofold. First, it has the ability to quantitatively state the uncertainty associated with a particular IUH form. Second, the IUH parameters can be determined from the information specified for its derivation. It goes without saying that it will be difficult to derive the general form using the linear systems theory. The other objective is to test the general IUH equation using observed rainfall-runoff data. Before applying the entropy theory, the concept of entropy is briefly described first, and then the entropy theory is outlined.

CONCEPT OF ENTROPY

In 1948, Shannon introduced the concept of information-theoretic or informational entropy, which is now more frequently referred to as Shannon entropy. Realizing that when information is specified, uncertainty is reduced or removed, a measure of uncertainty can be formulated as entropy. For a probability distribution $P = \{p_1, p_2, \dots, p_N\}$, where p_1, p_2, \dots, p_N are probabilities of N outcomes of a random experiment or values $(x_i, i = 1, 2, \dots, N)$ of a random variable X , this measure, denoted H , as a function of probabilities can be formulated as (Shannon, 1948):

$$H(p_1, p_2, \dots, p_N) = - \sum_{i=1}^N p_i \log_b p_i, \quad \sum_{i=1}^N p_i = 1 \quad (2)$$

where b is the base of the logarithm. Equation 2 satisfies a number of desiderata, such as continuity, additivity, symme-

try, monotonicity, expansibility, and others. This measure considers only the possibility of occurrence of an event, not its meaning or value. From a physical point of view, this can be a limitation of the entropy concept (Marchand, 1972). Equation 2 would lead to the maximum entropy if all probabilities are equal ($p_i = p, i = 1, 2, \dots, N$). The maximum entropy can be considered as a measure of complete uncertainty, i.e., the statistically most probable state would correspond to the maximum entropy.

In the search for an appropriate probability distribution for a given random variable, entropy should be maximized. In practice, however, it is common that some information is available on the random variable. The chosen probability distribution should then be consistent with the given information. There can be more than one distribution consistent with the given information. From all such distributions, one should choose the distribution that has the highest entropy. To that end, Jaynes (1957) formulated the principle of maximum entropy (POME) a full account of which is presented in a treatise by Levine and Tribus (1978). According to POME, the minimally prejudiced assignment of probabilities is that which maximizes entropy subject to the given information, i.e., POME takes into account all of the given information and at the same time avoids consideration of any information that is not given. Thus, the concept of entropy and principle of maximum entropy constitute what can be referred to as the entropy theory.

Since its advent, there has been a proliferation in applications of the entropy theory. The real impetus to entropy-based modeling in hydrology was, however, provided in the early 1970s (Amorocho and Espildora, 1973), and a great variety of entropy-based applications have since been reported and new applications continue to unfold. A historical perspective on entropy applications in environmental and water resources is given by Singh and Fiorentino (1992) and Singh (1997), and Harmancioglu and Singh (1998) discussed the use of entropy in water resources.

DERIVATION OF IUH USING THE ENTROPY THEORY

For deriving the instantaneous unit hydrograph (IUH), the entropy theory is comprised of the following steps: (1) defining the travel time as a random variable; (2) expressing the Shannon entropy for travel time; (3) specifying information on the travel time in terms of constraints; (4) using the principle of maximum entropy, maximizing the Shannon entropy subject to these constraints, and obtaining the least-biased probability density function (PDF) for travel time; (5) denoting the PDF of travel time as the IUH; (6) determining parameters of the IUH in terms of the specified constraints; and (7) expressing the entropy of the derived IUH equation. Before discussing these steps, it is worthwhile to clarify that the entropy theory is a statistical concept and therefore requires data. However, the data requirement is limited in that the theory needs only the constraints that can be obtained from data. The theory as such does not invoke any laws of fluid mechanics, such as momentum and energy conservation, although indirectly they can be reflected through constraints. This means that the entropy based method cannot be considered a physically based method in a fluid mechanical sense.

1. DEFINITION OF RANDOM VARIABLE

For a given rainfall event, runoff Q (L/T) can be expressed as a function of the area contributing it. It takes time for the flow contributed by any area to appear at the outlet, as the water has to travel from that area to the outlet. Thus, one may write Q as a function of area or the distance or length of travel, since length and area are related. Since the travel time uniquely depends on the travel distance (i.e., there is a unique relation between time of travel and length of travel), one can also express Q simply as a function of time, wherein parameters in the function will reflect the effect of different factors affecting runoff, such as slope, roughness, vegetation, microtopography, soil and infiltration characteristics, and rainfall characteristics. In a similar manner, one can also state that the IUH is a function of time, although realistically it also depends on rainfall-excess characteristics (space-time distribution), antecedent soil moisture, infiltration, and watershed characteristics.

In any given watershed, however small it is, there is virtually an infinite number of points where rainfall lands and water is generated. Thus, there is an infinite number of travel distances from points of landing, and consequently an infinite number of travel times. In other words, there is an infinite population of travel times. The travel time values do not follow any consistent pattern, and therefore travel time can be assumed as a random variable. Once this premise is accepted, one can employ the entropy theory to derive the probability distribution of travel time (t) depending on the information given. It turns out that IUH $h(t)$ has the characteristics of a probability distribution.

2. SHANNON ENTROPY

Since $h(t)$ is like a probability density function of travel time (t), the Shannon entropy (Shannon, 1948) of $h(t)$ can be written as:

$$H(t) = -\int_0^{\infty} h(t) \ln[h(t)] dt \quad (3)$$

The objective is to determine $h(t)$ by maximizing $H(t)$ given by equation 3, subject to specified constraints, in accordance with the principle of maximum entropy (POME). Maximization of entropy implies that $h(t)$ so derived will be least unbiased toward the information that is not specified or missing information and most biased toward the specified information. Thus, $h(t)$ can be derived as discussed in what follows.

3. SPECIFICATION OF CONSTRAINTS

It is clear that $h(t)$ must satisfy:

$$\int_0^{\infty} h(t) dt = 1 \quad (4)$$

because $h(t)$ is a probability density function and must satisfy the total probability theorem. Equation 4 ensures that the area under the IUH is unity or the amount of surface runoff represented by the IUH is unity.

For maximizing the entropy of the IUH, $H(t)$, one can hypothesize constraints in a general manner that $h(t)$ must satisfy:

$$\int_0^{\infty} g_r(t) h(t) dt = \overline{g_r(t)}, \quad r = 1, 2, \dots, R \quad (5)$$

where $g_r(t)$, $r = 1, 2, \dots, R$, are some functions of travel time, and R specifies the number of constraints. The right side of equation 5 denotes the expectation of $g_r(t)$. If, for example, $g_1(t) = t$, then equation 5 would denote the average travel time. If $g_2(t) = (t - \bar{t})^2$, then equation 5 would correspond to the variance of the travel time. In this manner, functions $g_r(t)$ can be defined in a meaningful way. Usually no more than two or three constraints are not needed. For practical purposes, these functions can be expressed in simple forms as:

$$\int_0^{\infty} \ln t h(t) dt = \overline{\ln t} \quad (6)$$

$$\int_0^{\infty} t^c h(t) dt = \overline{t^c} \quad (7)$$

where c is an exponent and an empirical parameter but can be related to the hydraulics of flow. Equation 6 expresses the expectation of the log values of travel times or the mean travel time in the logarithmic domain, whereas equation 7 expresses the moment of travel time raised to the power c . If, for example, c equals 1, then equation 7 expresses the mean travel time or the first moment about the origin; if c equals 2, then it expresses the second moment about the origin (equal to the variance of travel time minus the square of mean travel time); and so on.

4. MAXIMIZATION OF ENTROPY

Maximization of $H(t)$ given, by equation 3, can be achieved by using the principle of maximum entropy (POME) and the method of Lagrange multipliers. To that end, the Lagrangean function LF can be formulated as:

$$LF = -\int_0^{\infty} h(t) \ln h(t) dt - (\lambda_0 - 1) \left[\int_0^{\infty} h(t) dt - 1 \right] - \lambda_1 \left[\int_0^{\infty} \ln(t) h(t) dt - \overline{\ln t} \right] - \lambda_2 \left[\int_0^{\infty} t^c h(t) dt - \overline{t^c} \right] \quad (8)$$

where λ_0 , λ_1 , and λ_2 , are Lagrange multipliers that are associated with constraints defined by equations 4, 6, and 7. In order to obtain $h(t)$ that maximizes LF , one may recall the Euler-Lagrange equation of the calculus of variation; therefore, one differentiates LF with respect to $h(t)$ (noting t as a parameter and h as a variable) and equates the derivative to zero and obtains:

$$h(t) = \exp[-\lambda_0 - \lambda_1 \ln t - \lambda_2 t^c] \quad (9)$$

Inserting equation 9 into equation 4 one gets:

$$\begin{aligned} \exp(\lambda_0) &= \int_0^{\infty} \exp(-\lambda_1 \ln t - \lambda_2 t^c) dt \\ &= \int_0^{\infty} t^{-\lambda_1} \exp(-\lambda_2 t^c) dt \end{aligned} \quad (10)$$

Equation 10 can be expressed in terms of the gamma function as follows. Let $x = \lambda_2 t^c$. Then $t = (x/\lambda_2)^{1/c}$ and $dt = dx/[c\lambda_2^{1/c}x^{(c-1)/c}]$. Thus, equation 10 can be written as:

$$\exp(\lambda_0) = \frac{\lambda_2^{(1-\lambda_1)/c}}{c} \int_0^\infty x^{(1-\lambda_1)/c-1} \exp(-x) dx \quad (11)$$

The integral term in equation 11 is the gamma function, and therefore one gets:

$$\exp(\lambda_0) = \frac{\lambda_2^{(1-\lambda_1)/c}}{c} \Gamma\left(\frac{1-\lambda_1}{c}\right) \quad (12)$$

5. PDF OF TRAVEL TIME AS THE IUH

Substitution of equation 12 into equation 9 yields:

$$\begin{aligned} h(t) &= \frac{c\lambda_2^{(1-\lambda_1)/c}}{\Gamma\left(\frac{1-\lambda_1}{c}\right)} \exp(-\lambda_1 \ln t - \lambda_2 t^c) \\ &= \frac{c\lambda_2^{(1-\lambda_1)/c}}{\Gamma\left(\frac{1-\lambda_1}{c}\right)} t^{-\lambda_1} \exp(-\lambda_2 t^c) \end{aligned} \quad (13)$$

Equation 13 can be considered as a general equation of the IUH expressed by $h(t)$. Using the Maxwell-Boltzmann statistic, Lienhard (1964, 1972) derived a generalized probability distribution, which is similar to equation 13, but his procedure is much more complicated. Equation 13 has three parameters: λ_1 , λ_2 , and c . Exponent c can be either specified or determined by trial and error, or it can be estimated using the entropy method. Parameters λ_1 , λ_2 and c are now determined using the entropy theory.

6. DETERMINATION OF PARAMETERS

First, the Lagrange parameters λ_1 and λ_2 are determined as follows. Taking the logarithm of equation 12, one obtains:

$$\lambda_0 = \frac{\lambda_1 - 1}{c} \ln \lambda_2 - \ln c + \ln \left[\Gamma\left(\frac{1-\lambda_1}{c}\right) \right] \quad (14)$$

Differentiating equation 14 with respect to λ_1 yields:

$$\frac{\partial \lambda_0}{\partial \lambda_1} = \frac{\ln \lambda_2}{c} - \frac{1}{\Gamma[(1-\lambda_1)/c]} \frac{\partial \Gamma[(1-\lambda_1)/c]}{\partial \lambda_1} \quad (15)$$

Likewise, differentiation of equation 14 with respect to λ_2 yields:

$$\frac{\partial \lambda_0}{\partial \lambda_2} = \frac{\lambda_1 - 1}{c\lambda_2} \quad (16)$$

On the other hand, differentiation of equation 10 with respect to λ_1 leads to:

$$\frac{\partial \lambda_0}{\partial \lambda_1} = - \frac{\int_0^\infty \ln t \exp(-\lambda_1 \ln t - \lambda_2 t^c) dt}{\int_0^\infty \exp(-\lambda_1 \ln t - \lambda_2 t^c) dt} \quad (17)$$

Multiplying and dividing equation 17 by $\exp(-\lambda_0)$ and making use of equations 4, 6, and 9, one gets:

$$\frac{\partial \lambda_0}{\partial \lambda_1} = - \frac{\int_0^\infty \ln t \exp(-\lambda_0 - \lambda_1 \ln t - \lambda_2 t^c) dt}{\int_0^\infty \exp(-\lambda_0 - \lambda_1 \ln t - \lambda_2 t^c) dt} = -\overline{\ln t} \quad (18)$$

In a similar manner, differentiation of equation 10 with respect to λ_2 and making use of equations 4, 6, and 7 lead to:

$$\frac{\partial \lambda_0}{\partial \lambda_2} = - \frac{\int_0^\infty t^c \exp(-\lambda_0 - \lambda_1 \ln t - \lambda_2 t^c) dt}{\int_0^\infty \exp(-\lambda_0 - \lambda_1 \ln t - \lambda_2 t^c) dt} = -\overline{t^c} \quad (19)$$

Equating equation 15 to equation 18 results in:

$$\frac{\ln \lambda_2}{c} - \frac{1}{\Gamma[(1-\lambda_1)/c]} \frac{\partial \Gamma[(1-\lambda_1)/c]}{\partial \lambda_1} = -\overline{\ln t} \quad (20)$$

Similarly, equating equation 16 to equation 19 one obtains:

$$\frac{\lambda_1 - 1}{c\lambda_2} = -\overline{t^c} \quad (21)$$

Now, differentiation of equation 16 with respect to λ_2 yields:

$$\frac{\partial^2 \lambda_0}{\partial \lambda_2^2} = - \frac{\lambda_1 - 1}{c\lambda_2^2} \quad (22)$$

Likewise, differentiation of equation 19 with respect to λ_2 and making use of equations 4 and 5 lead to:

$$\frac{\partial \lambda_0}{\partial \lambda_2} = \frac{\int_0^\infty (t^c)^2 \exp(-\lambda_0 - \lambda_1 \ln t - \lambda_2 t^c) dt}{\int_0^\infty \exp(-\lambda_0 - \lambda_1 \ln t - \lambda_2 t^c) dt} = \overline{(t^c)^2} \quad (23)$$

Equating equation 22 to equation 23 one obtains:

$$\frac{\lambda_1 - 1}{c\lambda_2^2} = -\overline{(t^c)^2} \quad (24)$$

Equations 20, 21, and 24 constitute a system of three non-linear equations that are used to solve for the Lagrange parameters λ_1 , λ_2 , and λ_0 for given constraints and exponent c .

7. ENTROPY OF THE IUH

Substitution of equation 13 into equation 3 yields the IUH entropy:

$$\begin{aligned} H &= -\ln c - \frac{1-\lambda_1}{c} \ln \lambda_2 + \ln \Gamma\left(\frac{1-\lambda_1}{c}\right) \\ &\quad + \lambda_1 \overline{\ln t} + \lambda_2 \overline{t^c} \end{aligned} \quad (25)$$

where the Lagrange parameters λ_1 and λ_2 are as determined earlier. If equation 21 is inserted into equation 25, then the IUH entropy is expressed only in terms of parameters λ_1 and c as:

$$H = -\ln c - \frac{1-\lambda_1}{c} \ln \left[\frac{1-\lambda_1}{ct^c} \right] + \ln \Gamma \left(\frac{1-\lambda_1}{c} \right) + \lambda_1 \overline{\ln t} + \frac{\lambda_1 - 1}{c} \quad (26)$$

A higher value of entropy means more ignorance or uncertainty, meaning more information will be needed to characterize the IUH.

SPECIAL CASES

It may be interesting to derive special cases of equation 13, which have been popular for either representing the IUH or statistical analyses in hydrology, such as flood frequency analyses, drought characterization, and reliability analysis.

Gamma IUH

Let $c = 1$, $\lambda_2 = 1/k$, and $(c-\lambda_1)/c = n$. Equation 13 becomes:

$$h(t) = \frac{1}{k\Gamma(n)} \left(\frac{t}{k} \right)^{n-1} \exp \left[-\left(\frac{t}{k} \right) \right] \quad (27)$$

Equation 27 is the gamma distribution derived by Nash (1957, 1958) by representing a watershed as a cascade of linear reservoirs and is perhaps the most popular form of the IUH used in hydrology. Here $h(t)$ is the IUH of the cascade, n represents the number of reservoirs in the cascade where $\Gamma(n)$ is the gamma function of n , k represents the lag time of a reservoir, and t is time. Note that the lag time of the watershed will be nk , which can be shown to be equal to the time difference between the centroid of rainfall excess and the centroid of surface runoff.

It may be noted that if the travel time is measured as $t^* = t - t_0$, where t_0 is some initial time. Equation 27 then becomes:

$$h(t) = \frac{1}{k\Gamma(n)} \left(\frac{t-t_0}{k} \right)^{n-1} \exp \left[-\left(\frac{t-t_0}{k} \right) \right] \quad (28)$$

Equation 28 is the three-parameter Pearson distribution, which is frequently used in flood frequency analysis. In a similar manner, if $x = \log t$, then:

$$h(t) = \frac{1}{kt\Gamma(n)} \left[\frac{\ln(t-t_0)}{k} \right]^{n-1} \exp \left[-\left(\frac{\ln(t-t_0)}{k} \right) \right] \quad (29)$$

Equation 29 is the log-Pearson type 3 distribution, which is also commonly used in flood frequency analysis.

Lienhard IUH

Let $c = 2$, $\lambda_2 = (m+1)/(2k^2)$, and $\lambda_1 = -m$. Here, k is the mean travel or residence time. Equation 13 becomes:

$$h(t) = \frac{2}{k\Gamma\left(\frac{m+1}{2}\right)} \left(\frac{m+1}{2} \right)^{(m+1)/2} \times \left(\frac{t}{k} \right)^m \exp \left[-\frac{m+1}{2} \left(\frac{t}{k} \right)^2 \right] \quad (30)$$

Equation 30 is the Lienhard equation, which Lienhard (1964) derived using the Boltzmann statistics as a generalized gamma probability distribution. Using observed rainfall-runoff data, he found that equation 30 with $m = 2$ would be adequate. Thus, equation 30 becomes:

$$h(t) = \frac{1}{k\Gamma\left(\frac{3}{2}\right)} \left(\frac{3}{2} \right)^{3/2} \left(\frac{t}{k} \right)^2 \exp \left[-\frac{3}{2} \left(\frac{t}{k} \right)^2 \right] \quad (31)$$

Nakagami-m Distribution

Let $c = 2$, $\lambda_2 = (a/b)$, and $\lambda_1 = 1-2a$. Equation 13 becomes:

$$h(t) = \frac{2}{\Gamma(a)} \left(\frac{a}{b} \right)^a t^{2a-1} \exp \left[-\frac{a}{b} t^2 \right] \quad (32)$$

Equation 32 is the Nakagami- m distribution function. It is a slightly different form of equation 30 and has received some attention in recent years (Rai et al., 2010). Equation 32 is another form of the IUH equation. It can be noted that parameters a and b can be expressed as:

$$a = \frac{E^2[t^2]}{\sigma[t^2]}; \quad a \geq 1/2$$

$$b = E[t^2] \quad (33)$$

where b is a shape parameter that controls the spread, σ is the variance, and E is the expectation operator. If t has a Nakagami distribution with parameters a and b , then t^2 has a gamma distribution with shape parameter a and scale parameter b/a .

OTHER FORMS

As an aside, equation 13 specializes into the following forms, which have been employed for frequency analyses of hydrologic extremes and reliability analysis.

Exponential Distribution

Let $c = 1$, $\lambda_1 = 0$, and $\lambda_2 = 1/k$, where k is the mean travel time. Equation 13 becomes:

$$h(t) = \frac{1}{k} \exp \left(-\frac{t}{k} \right) \quad (34)$$

Equation 34 is employed for modeling time interval between extremes that are assumed to follow the Poisson process.

Weibull Distribution

Let $\lambda_1 = 1-c$ and $\lambda_2 = (1/k^c)$. Then equation 13 becomes:

$$h(t) = \frac{c}{k} \left(\frac{t}{k} \right)^{c-1} \exp \left[-\left(\frac{t}{k} \right)^c \right] \quad (35)$$

which is the Weibull distribution, which is frequently used for representing the rainfall hyetograph.

Raleigh Distribution

Let $c = 2$, $\lambda_1 = -1$, and $\lambda_2 = (1/k^c)$. Then equation 13 becomes:

$$h(t) = \frac{2}{k} \left(\frac{t}{k} \right) \exp \left[-\left(\frac{t}{k} \right)^2 \right] \quad (36)$$

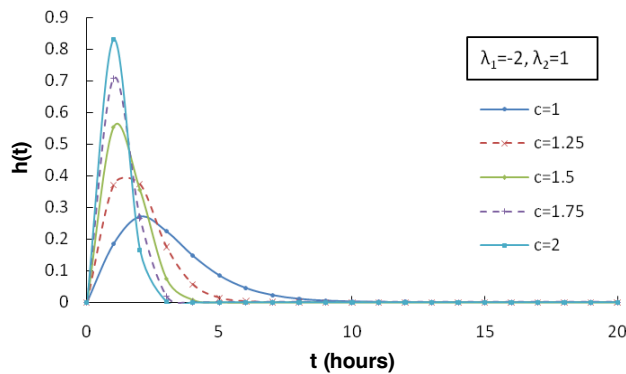


Figure 1. Change in IUH with change in c .

Equation 36 is the Raleigh distribution and is used in reliability analysis.

Maxwell Molecular Velocity Distribution

Let $c = 2$, $\lambda_2 = (1/k^2)$, and $\lambda_1 = 0$. Then equation 13 becomes:

$$h(t) = \frac{1}{k} \frac{2}{\sqrt{\pi}} \exp \left[- \left(\frac{t}{k} \right)^2 \right] \quad (37)$$

Equation 37 is the Maxwell molecular speed distribution, which is used in quantum physics.

CHARACTERISTICS OF THE GENERAL IUH EQUATION

The general IUH equation has three parameters: c , λ_1 , and λ_2 . To evaluate the change in the IUH shape with changes in these parameters, $h(t)$ was computed by changing one parameter at a time while keeping the other parameters constant and then plotting the results. The ranges of parameter values selected were based on the surface runoff hydrographs produced. Figure 1 shows the variation in $h(t)$ with changes in c , figure 2 with changes in λ_1 , and figure 3 with changes in λ_2 . It is seen that a high value of c leads to a high value of the IUH peak, and a low value of c leads to a low value of the IUH peak. The reverse is true for the IUH base period. In general, the IUH is quite sensitive to c because the travel time is to the power of c . The value of λ_1 was found to be negative. The IUH is sensitive to the change in λ_1 , with smaller negative values producing higher IUH peaks, and vice versa. On the other hand, larger values of λ_2 produce higher IUH peaks, and vice versa.

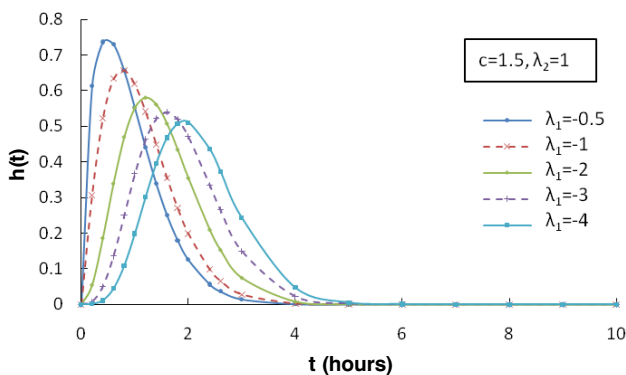


Figure 2. Change in IUH with change in λ_1 .

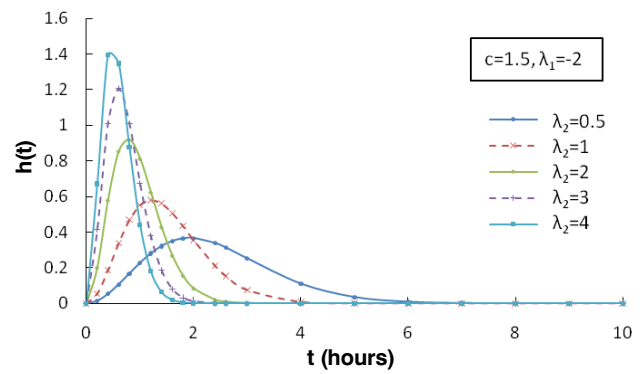


Figure 3. Change in IUH with change in λ_2 .

TESTING THE GENERAL IUH EQUATION

The general IUH equation was tested on two watersheds (W-3 and W-8) located near Hastings, Nebraska. These are small experimental agricultural watersheds maintained by the USDA Agricultural Research Service in cooperation with the Nebraska Agricultural Experiment Station. As shown in figure 4, watershed W-3 is 199.46 ha (481 acres) or 1.92 km² (0.751 square miles) in area, and watershed W-8 is 844.2 ha (2,086 acres) or 8.44 km² (3.26 square miles) in area. Land use and cropping patterns of these watersheds changed from year to year. Since these were agricultural watersheds, they contained crops. Seven rainfall-runoff events during 1960-1961, 1964-1965, and 1966-1967 as reported in *Hydrologic Data for Experimental Agricultural Watersheds in the United States* (USDA, 1960-61, 1964-65, 1966-67) were selected for the two watersheds: four for W-3 and three for W-8. These events are summarized in table 1. These watersheds were selected for the easy availability of data. It should be emphasized that the entropy-based IUH does not have any parameters that need to be calibrated. Therefore, all seven events were used for testing the entropy method. This method was also compared with the method based on the gamma IUH, where parameters need to be calculated for each event.

ESTIMATION OF LAGRANGE MULTIPLIERS AND GAMMA IUH PARAMETERS

The general IUH derived using entropy is based on two constraints, which are based on the travel time. To that end, the time of concentration (t_c) was first determined using the Kirpich (1940) formula:

$$t_c = 0.0078L^{0.77}S^{-0.385} \quad (38)$$

Table 1. Rainfall-runoff data for watersheds W-3 and W-8.

Event Date	Rainfall Amount (cm)	Rainfall Duration (min)	Runoff Amount (cm)	Infiltration Amount (cm)
Watershed W-3				
15 May 1960	5.639	48	2.168	3.471
15 August 1961	4.166	28	0.646	3.520
29 June 1965	1.829	12	1.014	0.815
8 July 1967	3.200	12	0.438	2.762
Watershed W-8				
15 May 1960	5.472	18	1.521	3.951
1 June 1965	3.708	58	1622	2.086
8 July 1967	3.708	14	0.673	3.035

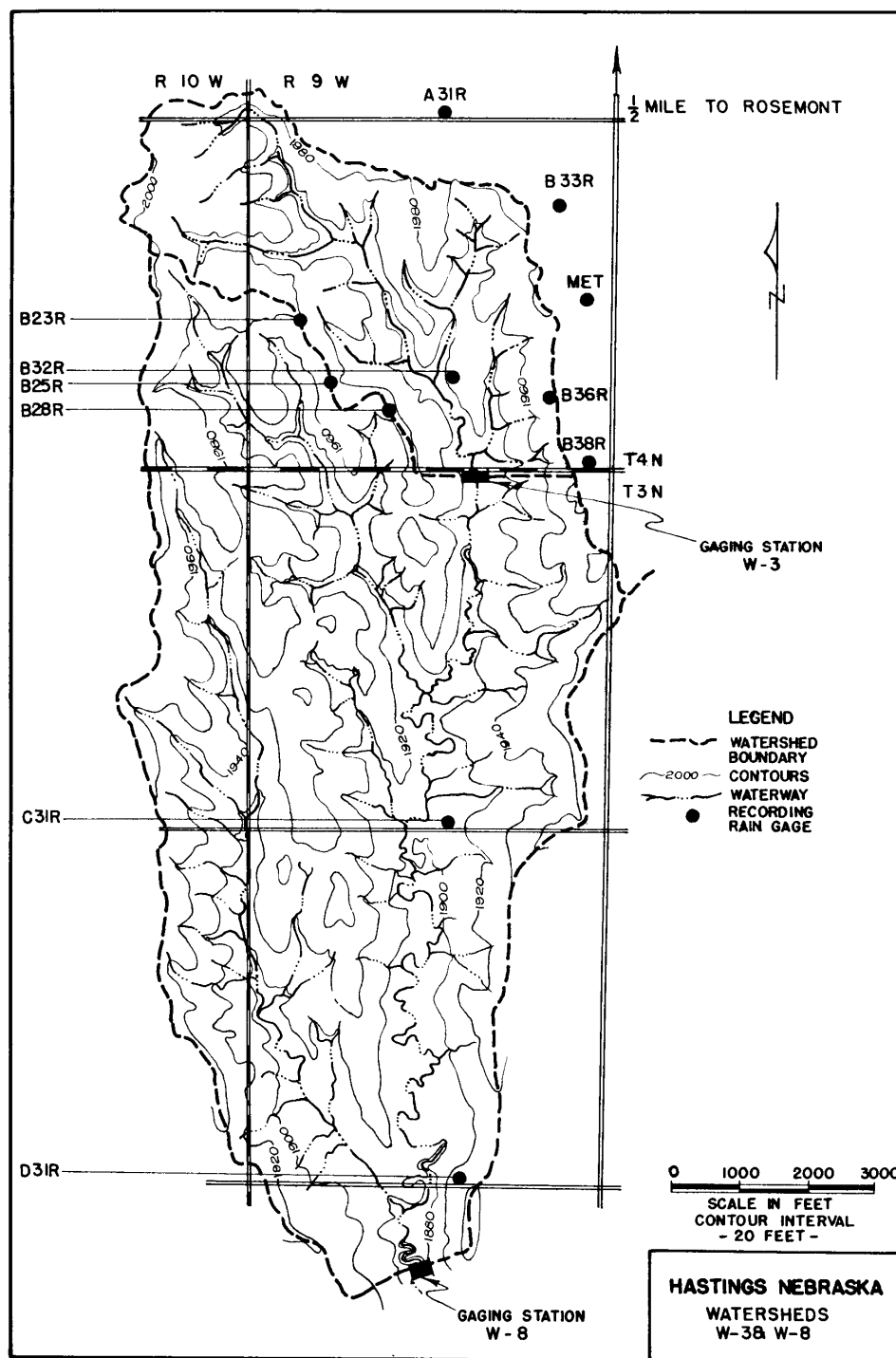


Figure 4. Watersheds W-3 and W-5 near Hastings, Nebraska.

where L is the channel length from head water to the outlet in feet, S is the average watershed slope, and t_c is given in minutes. For watershed W-3, $L = 7800$ ft (2377.44 m), $S = 0.0077$, and t_c was 50.43 min; for watershed W-8, $L = 16600$ ft (5059.68 m), $S = 0.0071$, and t_c was 90.22 min. Second, a large number of points on the watershed surface were selected, and the distances along flow paths from these points to the outlet were noted. The slopes of these flow paths were computed because the watershed maps had elevation contours. It was then hypothesized that the travel time from any point to the outlet will be a fraction of the time of concentration in proportion to its distance and

slope. In other words, equation 38 would apply. In this manner, about 30 points were selected for each watershed and their travel times were computed.

Equation 13 has three parameters (λ_1 , λ_2 , and c), which were determined using equations 20, 21, and 24 in concert with constraint equations 6 and 7. These equations were solved numerically and the travel time values obtained earlier were employed. The parameters so obtained were: $c = 1.26$, $\lambda_1 = -0.34$, and $\lambda_2 = 1.61$ for watershed W-3, and $c = 1.56$, $\lambda_1 = -0.54$, and $\lambda_2 = 0.52$ for watershed W-8. With these parameters, the IUH was computed using equation 13.

Table 2. Parameters n and k of gamma IUH for rainfall-runoff data for watersheds W-3 and W-8.

Event Date	Uniform Infiltration Capacity Rate (cm h ⁻¹)		Non-Uniform Infiltration Capacity Rate (cm h ⁻¹)	
	n	k	n	k
Watershed W-3				
15 May 1960	0.837	0.92	--	--
15 August 1961	0.824	0.89	--	--
29 June 1965	0.783	0.94	0.792	0.91
8 July 1967	0.815	1.00	0.811	0.96
Watershed W-8				
15 May 1960	2.27	1.02	2.25	1.01
1 June 1965	2.38	1.36	--	--
8 July 1967	2.38	1.36	2.41	1.27

Because the gamma distribution is frequently used as an IUH, it was decided to use it for comparison purposes. For the exponent value $c = 1$, the general UH equation (eq. 13) would reduce to a gamma distribution given by equation 27. Parameters of the gamma distribution representing the IUH were estimated by the method of moments, which is commonly employed in surface water hydrology (Nash, 1957; Dooge, 1973; Singh 1988). In this method, the first two moments of travel time are computed, and then parameters n and k of the gamma distribution are determined. This yields $\bar{t} = nk$ and $\bar{t}^2 = nk^2(n+1)$. However, this required computation of the infiltration capacity rate. For purposes of simplicity, the rate of infiltration capacity was assumed uniform during the period of a rainfall event and the rainfall excess hyetograph was hence computed. The effect of this assumption will be examined later. For all events of watersheds W-3 and W-8, the values of parameters n and k were obtained, as given in table 2.

HYDROGRAPH PREDICTION

Next, with the IUH obtained using equation 13 (henceforth referred to as the entropy method) and using equation 27 (henceforth referred to as the gamma IUH method),

the surface runoff hydrograph was computed using equation 1 or 2 by convoluting this IUH with rainfall excess and compared with the observed surface runoff hydrograph for each of the seven rainfall-runoff events. It is recognized that the rainfall-excess with uniform infiltration capacity rate was computed in a crude way, but the objective here was to qualitatively test the entropy based IUH equation. Errors in runoff peak and time to peak for both methods were computed, as given in table 3. The entropy method produced surface runoff hydrographs that were in reasonable agreement with observed surface runoff hydrographs. For two sample events, observed and computed surface runoff hydrographs are shown in figures 5 and 6. The error with the entropy method was 11% to 27% in peak runoff and 4% to 17% in time to peak, considering all the events for watershed W-3. For watershed W-8, the error was 6.2% to 13.5% in peak runoff and 9.3% to 124% in time to peak. Excepting time to peak for one event, the accuracy of the entropy method in predicting surface runoff hydrograph was remarkable considering the fact that no fitting was involved.

For the gamma IUH with parameters estimated for each event, surface runoff hydrographs were computed, and errors in peak runoff and time to peak are shown in table 4. Although the gamma IUH method produced surface runoff hydrographs that looked reasonably good, it was not as accurate as the entropy method, even though the gamma IUH parameters were computed for each event. The error was 19% to 39% in peak runoff and 4.3% to 114% in time to peak for watershed W-3, and the error was 4.6% to 20% in peak and 17% to 124% in time to peak on watershed W-8. Comparing the two methods, the entropy value was 1.50 nats for the entropy method (using eq. 26) and 0.318 to 0.361 nats for the gamma IUH method (using eqs. 3 and 27). Thus, it is seen that the entropy method better produced the IUH.

EFFECT OF INFILTRATION ON SURFACE RUNOFF HYDROGRAPH

In order to evaluate the effect of infiltration on the computed surface runoff hydrograph, infiltration was computed

Table 3. Errors in runoff peak characteristics with uniform infiltration capacity rate.

Event Date	Method Used	Observed Peak (cm h ⁻¹)	Computed Peak (cm h ⁻¹)	Error in Peak (%)	Observed Time to Peak (min)	Computed Time to Peak (min)	Error in Time to Peak (%)	Entropy (nats)
Watershed W-3								
8 July 1967	Entropy	0.318	0.415	17.5	24	20	16.7	1.11
	Moment		0.442	39		20	16.7	0.318
29 June 1965	Entropy	1.026	0.791	22.9	55	60	9.1	1.11
	Moment		0.765	25.4		48	12.7	0.352
15 May 1960	Entropy	2.367	1.738	26.6	46	48	4.3	1.11
	Moment		1.686	28.8		48	4.3	0.361
15 August 1961	Entropy	0.366	0.406	10.9	28	48	71.4	1.11
	Moment		0.437	19.3		60	114.3	0.339
Watershed W-8								
8 July 1967	Entropy	0.172	0.186	8.1	110	120	9.3	1.5
	Moment		0.18	4.6		130	18.6	1.18
15 May 1960	Entropy	0.676	0.585	13.5	104	120	15.6	1.5
	Moment		0.536	20.7		150	44.5	1.18
1 June 1965	Entropy		0.424	6.19		130	124	1.5
	Moment	0.452	0.541	19.7	58	110	89.7	3.58

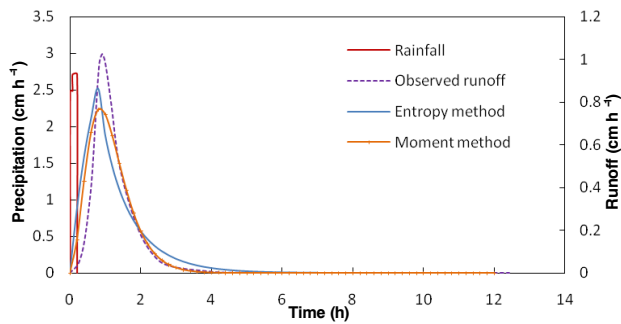


Figure 5. Comparison of surface runoff hydrographs computed using the entropy method and the gamma IUH method for rainfall-runoff for event of 29 June 1965 on watershed W-3 near Hasting, Nebraska.

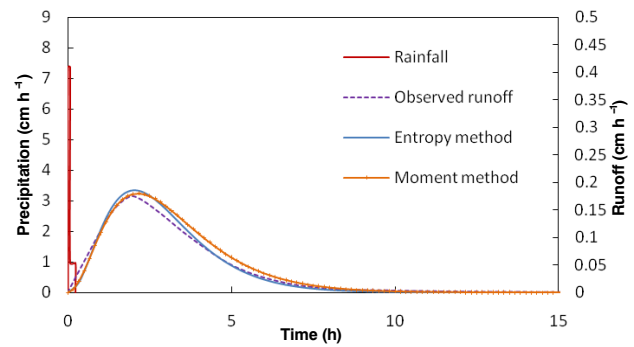


Figure 6. Comparison of surface runoff hydrographs computed using the entropy method and the gamma IUH method for rainfall-runoff for event of 8 July 1967 on watershed W-8 near Hasting, Nebraska.

Table 4. Errors in runoff peak characteristics with non-uniform infiltration.

Event Date	Method Used	Observed Peak (cm h ⁻¹)	Computed Peak (cm h ⁻¹)	Error in Peak (%)	Observed Time to Peak (min)	Computed Time to Peak (min)	Error in Time to Peak (%)	Entropy (nats)
Watershed W-3								
8 July 1967	Entropy	0.318	0.415	17.5	24	20	16.7	1.11
	Moment		0.445	39		20	17.5	0.31
29 June 1965	Entropy	1.026	0.864	15.8	55	68	9.1	1.11
	Moment		0.773	25.4		42	23.6	0.346
Watershed W-8								
8 July 1967	Entropy	0.172	0.166	3.5	110	120	9.1	1.5
	Moment		0.183	4.6		130	18.6	3.53
15 May 1960	Entropy	0.676	0.793	17.4	104	103	0.9	1.5
	Moment		0.576	14.8		123	18.3	2.76

in a slightly more non-uniform manner and then rainfall excess was determined. About 40% of rainfall was allowed to infiltrate in the beginning, and then infiltration was allowed to occur uniformly. The surface runoff hydrograph was then computed using both the entropy and gamma IUH methods. Observed and computed surface runoff hydrographs are shown for two sample events in figures 7 and 8. The error with the entropy method was 15.8% to 25% in peak runoff and 9.1% to 17% in time to peak for watershed W-3. For the gamma IUH method, the error was 25% to 39% in peak runoff and 17.5% to 24% in time to peak. For watershed W-8, the error with the entropy method varied from 3.5% to 7.4% in peak runoff and from 0.9% to 9.1% in time to peak. For the gamma IUH method, the error was 4.6% to 15% in peak runoff and 18% to 19% in time to peak. A slightly more improved

way for accounting for infiltration led to a significant improvement in predicting the surface runoff hydrograph. Comparing the two methods, the entropy value was 1.5 nats for the entropy method and 2.53 to 2.76 nats for the gamma IUH method. Overall, the entropy method better produced the IUH.

Comparing the results obtained for the two watersheds, it was found that the runoff hydrograph and its characteristics were better predicted for watershed W-8 than for watershed W-3. One plausible explanation is that W-8 is significantly larger than W-3 and is therefore more linear time-invariant than W-3. This means that the IUH hypothesis would better apply to W-8 than to W-3 and hence result in more accurate predictions.

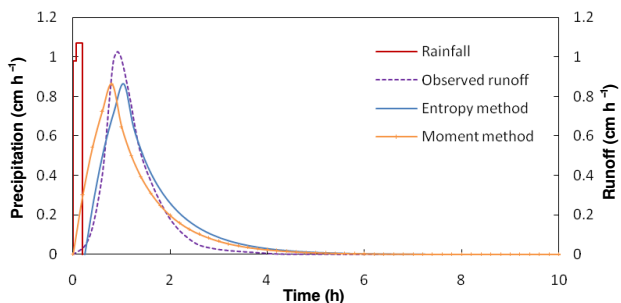


Figure 7. Comparison of surface runoff hydrographs computed using the entropy method and the gamma IUH method with improved infiltration for rainfall-runoff for event of 29 June 1965 on watershed W-3 near Hasting, Nebraska.

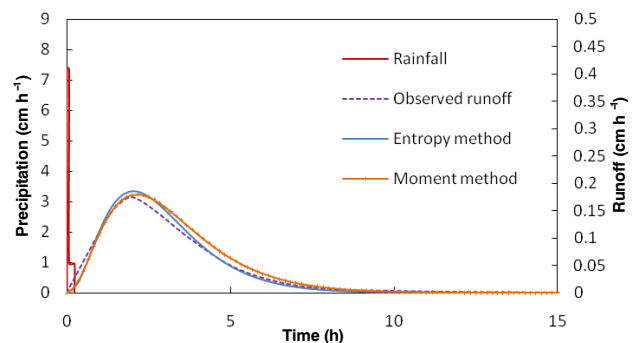


Figure 8. Comparison of surface runoff hydrographs computed using the entropy method and the gamma IUH method with improved infiltration for rainfall-runoff for event of 8 July 1967 on watershed W-8 near Hasting, Nebraska.

CONCLUSIONS

The following conclusions are drawn from this study:

- The entropy theory leads to a general equation for the instantaneous unit hydrograph. This general equation specializes into several popular IUH equations that have been reported in the hydrologic literature, and several other equations that have been found useful for analyses of hydrologic extremes and reliability analysis.
- For all events on two sample watersheds (W-3 and W-8), the entropy method is found superior to the gamma IUH method.
- The entropy-based IUH equation satisfactorily produces surface runoff hydrographs.
- The advantage of the entropy theory is that it permits derivation of the IUH based on the limited specified information alone. This information can be obtained from usually available data, such as slope, length, and area.

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